ON THE STABILITY OF MOTION OF AN UNPERTURBED PHYSICAL PENDULUM

(OB USTOICHIVOSTI DVIZHENIIA NEVOZMUSHCHAEMOGO FIZICHESKOGO MAIATNIKA)

PMM Vol.28, № 2, 1964, pp.364-366

D.M.KLIMOV
(Moscow)

(Received October 8, 1963)

The motion of a known [1] unperturbed physical pendulum is considered. The condition for stability of its motion (for the case when the pendulum suspension point moves along a circle on the surface of a fixed sphere with constant velocity) is established.

1. Let us consider the motion of a physical pendulum for which the point of suspension O moves along the surface of a fixed S of radius R surrounding the earth. Let us consider that the forces of attraction of the pendulum to earth reduce to the single force mg applied at the center of gravity G (Fig.1) and directed along the geocentric vertical (normal to the surface of the sphere). The distance OG is denoted by l .

Let us study the motion of the pendulum in a translationally moving $\xi\eta\zeta$ coordinate system with center at the suspension point O . Let us introduce the Darboux trihedron x_0, y_0, z_0 . Let us direct its x_0-axis along the vector $\mathbf{V}$ of the absolute velocity of the suspension point O , the z_0-axis along the normal to the sphere. Let us couple the x,y,z axes to the pendulum in such a manner that the direction of the z-axis would coincide with the direction of the line GO. The pendulum position relative to the Darboux trihedron is determined by the angles ψ, θ and φ (Fig.2).

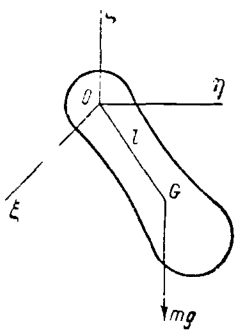

Fig. 1

Let us assume that the pendulum moments of inertia relative to the x and y-axes are the principal moments and equal mlR; the pendulum moment of inertia relative to the z-axis is denoted by C .

Such a pendulum is unperturbed, i.e. the z-axis is directed along the geocentric vertical [1] while its point of suspension moves arbitrarily over a fixed sphere S

Projections of the angular velocity $\mathbf{u}_0$ of the Darboux trihedron on its axes have the form

$$p_0 = 0, \quad q_0 = -\frac{V(t)}{R}, \quad r_0 = r_0(t) \qquad (1.1)$$

Here q_0 and r_0 are considered to be known functions of time. Evaluating the projections of the angular velocity of the trihedron $x_2y_2z_2$ on its axes (Fig. 2), we find

$$p_2 = q_0 \sin\psi \sin\theta - r_0 \cos\psi \sin\theta + \psi^{\cdot} \cos\theta$$
$$q_2 = q_0 \cos\psi + r_0 \sin\psi + \theta^{\cdot} \qquad (1.2)$$
$$r_2 = -q_0 \sin\psi \cos\theta + r_0 \cos\psi \cos\theta + \psi^{\cdot} \sin\theta$$

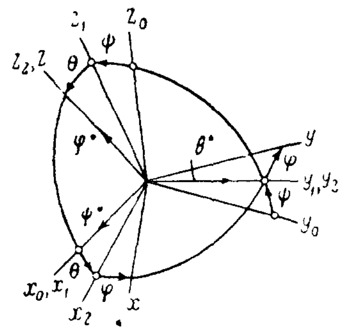

Fig. 2

Here and henceforth, the dot denotes differentiation with respect to time.

Let ω denote the vector of the pendulum angular velocity. We have

$$\omega_{x_2} = p_2, \qquad \omega_{y_2} = q_2, \qquad \omega_{z_2} = r_2 + \varphi^{\cdot} \qquad (1.3)$$

Let us write the equations of pendulum motion in the form

$$mlRp_2^{\cdot} + Cq_2 (r_2 + \varphi^{\cdot}) - mlRq_2r_2 = M_{x_2}$$
$$mlRq_2^{\cdot} + mlRp_2r_2 - Cp_2 (r_2 + \varphi^{\cdot}) = M_{y_2} \qquad (1.4)$$
$$[C (r_2 + \varphi^{\cdot})]^{\cdot} = M_{z_2}$$

Here M_{x_2}, M_{y_2}, and M_{z_2} are projections of the moments of the external forces among which are the forces of inertia of the translatory motion.

Performing simple computations and taking account of (1.1), we find

$$M_{x_2} = -mgl \sin\psi - mlRq_0r_0 \cos\psi + mlRq_0^2 \sin\psi, \qquad M_{z_2} = 0 \qquad (1.5)$$
$$M_{y_2} = -mgl \cos\psi \sin\theta + mlRq_0^{\cdot} \cos\theta + mlRq_0r_0 \sin\psi \sin\theta + mlRq_0^2 \cos\psi \sin\theta$$

2. The kinetic energy of the physical pendulum (Fig. 3) has the form

$$T = {}^1/_2 \Sigma m_i \mathbf{r}_i^{\cdot 2} = {}^1/_2 \Sigma m_i (\mathbf{R}^{\cdot} + \boldsymbol{\varrho}_i^{\cdot})^2$$

Since $\boldsymbol{\varrho}_i^{\cdot} = \omega \times \boldsymbol{\varrho}_i$, then denoting $OG = \mathbf{l}$, we have

$$T = {}^1/_2 mV^2 + m\mathbf{V} \cdot (\mathbf{u}_0 \times \mathbf{l}) + m\mathbf{V} \cdot [(\omega - \mathbf{u}_0) \times \mathbf{l}] + {}^1/_2 mlR (p_2^2 + q_2^2) + {}^1/_2 C (r_2 + \varphi^{\cdot})^2 \qquad (2.1)$$

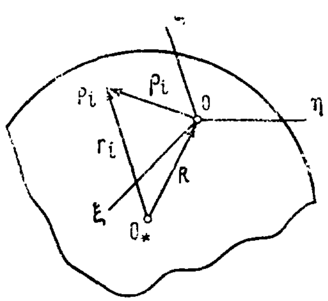

Fig. 3

Let us now assume that the pendulum suspension point moves along the arc of a circle on the surface of a fixed sphere with a constant velocity. In this case the angular velocity of the Darboux trihedron u_0 is constant and the kinetic energy of the pendulum is explicitly independent of the time.

Since the forces of attraction are potential forces, the generalized energy integral [2]

$$T_2 - T_0 + \Pi = h \qquad (2.2)$$

holds.

Here T_2 is the part of the kinetic energy which is a quadratic form of the generalized velocities, T_0 is the part of the kinetic energy independent of the generalized velocities, Π is the potential energy of the attractive force.

Taking account of (1.1) to (1.3) and (2.1), let us write the integral (2.2) in the form

$$W_1 = \tfrac{1}{2} mlR\,(\psi^{\cdot 2}\cos^2\theta + \theta^{\cdot 2}) + \tfrac{1}{2}C\,(\psi^{\cdot}\sin\theta + \varphi^{\cdot})^2 + \tag{2.3}$$
$$+ mlRq_0\,(q_0\cos\psi + r_0\sin\psi)\cos\theta - \tfrac{1}{2} mlR\,(q_0\sin\psi - r_0\cos\psi)^2\sin^2\theta -$$
$$- \tfrac{1}{2} mlR\,(q_0\cos\psi + r_0\sin\psi)^2 - \tfrac{1}{2} C\,(q_0\sin\psi - r_0\cos\psi)^2\cos^2\theta - mgl\cos\psi\cos\theta = h$$

From the third equation of the system (1.4), we obtain the other first integral

$$W_2 = C\,(r_0 + \varphi^{\cdot}) = H \tag{2.4}$$

3. Using (1.1) to (1.5), let us write the equations of pendulum motion for the case of u_0 = const . We have

$$\tag{3.1}$$

$$mlR\,(q_0\psi^{\cdot}\cos\psi\sin\theta + q_0\theta^{\cdot}\sin\psi\cos\theta + r_0\psi^{\cdot}\sin\psi\sin\theta - r_0\theta^{\cdot}\cos\psi\cos\theta +$$
$$+ \psi^{\cdot\cdot}\cos\theta - \psi^{\cdot}\theta^{\cdot}\sin\theta) + C\,(q_0\cos\psi + r_0\sin\psi + \theta^{\cdot})\,(-q_0\sin\psi\cos\theta +$$
$$+ r_0\cos\psi\cos\theta + \psi^{\cdot}\sin\theta + \varphi^{\cdot}) - mlR\,(q_0\cos\psi + r_0\sin\psi + \theta^{\cdot})\,(-q_0\sin\psi\cos\theta +$$
$$+ r_0\cos\psi\cos\theta + \psi^{\cdot}\sin\theta) = - mgl\sin\psi - mlR\,q_0r_0\cos\psi + mlRq_0^2\sin\psi$$

$$mlR\,(-q_0\psi^{\cdot}\sin\psi + r_0\psi^{\cdot}\cos\psi + \theta^{\cdot\cdot}) + mlR\,(q_0\sin\psi\sin\theta - r_0\cos\psi\sin\theta + \psi^{\cdot}\cos\theta)\times$$
$$\times(-q_0\sin\psi\cos\theta + r_0\cos\psi\cos\theta + \psi^{\cdot}\sin\theta) - C\,(q_0\sin\psi\sin\theta -$$
$$- r_0\cos\psi\sin\theta + \psi^{\cdot}\cos\theta)\,(-q_0\sin\psi\cos\theta + r_0\cos\psi\cos\theta + \psi^{\cdot}\sin\theta + \varphi^{\cdot}) =$$
$$= - mgl\cos\psi\sin\theta + mlRq_0r_0\sin\psi\sin\theta + mlRq_0^2\cos\psi\sin\theta$$

$$[C\,(-q_0\sin\psi\cos\theta + r_0\cos\psi\cos\theta + \psi^{\cdot}\sin\theta + \varphi^{\cdot})]^{\cdot} = 0$$

The system of nonlinear Equations (3.1) has the particular solution

$$\psi = 0, \quad \theta = 0, \quad \varphi^{\cdot} + r_0 = 0 \tag{3.2}$$

Let us investigate its stability. To do this, let us set in the perturbed motion

$$\psi = x_1, \quad \theta = x_2, \quad \psi^{\cdot} = x_3, \quad \theta^{\cdot} = x_4, \quad \varphi^{\cdot} = - r_0 + x_5$$

and let us consider the function $W = 2W_1 + 2r_0W_2$.

Expanding it in a power series in x_i (i = 1,2,..., 5), we obtain

$$W = W(0) + (mgl - Cq_0^2 - mlRr_0^2)\,x_1^2 + (mgl - mlRq_0^2 - mlRr_0^2)\,x_2^2 +$$
$$+ mlR\,(x_3^2 + x_4^2) + Cx_5^2 + \ldots$$

Here $W(0)$ is the value of the function W when $x_i = 0$ (i = 1,2,...,5); the higher order terms are denoted by dots.

The function $W - W(0)$ will be positive definite for sufficiently small x_i if

$$mgl - Cq_0^2 - mlRr_0^2 > 0, \qquad mgl - mlRq_0^2 - mlRr_0^2 > 0 \tag{3.3}$$

Since its derivative is zero by virtue of the equations of perturbed motion, the unperturbed motion (3.2) will be stable upon compliance with the inequalities (3.3).

If $C > mlR$, the inequalities (3.3) are replaced by the single condition

$$mgl > Cq_0^2 + mlRr_0^2$$

If $C \leqslant mlR$, the inequalities (3.3) reduce to the condition

$$u_0 < \nu \qquad (\nu^2 = g/R) \tag{3.4}$$

which agrees with the condition for stability of gyro-horizon compass [3].

It is not difficult to show that the condition (3.4) is necessary. To do this one should write the necessary conditions for the stability of the linearized system of equations (3.1) in the presence of small dissipative forces.

BIBLOGRAPHY

1. Ishlinskii, A.Iu., Ob otnositel'nom ravnovesii fizicheskogo maiatnika s podvizhnoi tochkoi opory (On the relative equilibrium of a physical pendulum with a fixed point of support). *PMM* Vol.20, № 3, 1956.

2. Bukhgol'ts, N.N., Osnovnoi kurs teoreticheskoi mekhaniki (Fundamental course in theoretical mechanics). ONTI NKTP SSSR, Part II, Moscow-Leningrad, 1937.

3. Zhbanov, Iu.K., Issledovanie svobodnykh kolebanii v sisteme avtonomnogo opredeleniia koordinat dvizhushchegosia ob'ekta (Investigation of free vibrations in a system of self-contained determination of the coordinates of a moving object). *PMM* Vol.24, № 6, 1960.

Translated by M.D.F.